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Out of Equilibrium Dynamics of the Toy Model with Mode Coupling and Trivial Hamiltonian[1]

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We extend our previous analysis of the toy model that mimics the mode coupling theory of supercooled liquids and glass transitions to the out of equilibrium dynamics. We derive a self-consistent set of equations for correlation and response functions.

KEY WORDS: Out of equilibrium; toy model; mode-coupling; glassy behavior

1. INTRODUCTION

Recently we have introduced a mean field toy model that mimics the mode coupling theory (MCT) of supercooled liquids and glass transitions with trivial Hamiltonian.[1,2,3] Analyses were limited to the equilibrium dynamics. An important feature of the model is that the strength of "hopping processes"[4] that destroys the non-ergodic state of the ideal MCT[5,6] can be tuned so that nonergodic state is still allowed in some region of the model parameter space. This implies that the so-called hopping processes do not seem to be the same as thermally activated processes.

In order to obtain further insights into the nature of MCT we consider the out of equilibrium dynamics of the model. In connection to this,

[1] This article is dedicated to Bob Dorfman on his 65th birthday. His persistent contributions to the kinetic theory and to the bases of nonequilibrium statistical physics over more than three decades taught us a great deal on the subjects and on the style of doing science.

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recently the out of equilibrium dynamics of the mean-field-type spin glass models and other related glassy models was considered.[7] In particular, the out of equilbrium dynamics of the spherical p-spin model,[8] represented by the closed set of equations for the off-equilibrium two time correlation function $C(t, t_w)$ and the response function $G(t, t_w)$, was analytically solved in the long time regime.[9] The analytic solution has revealed interesting features of out of equilibrium dynamics of the model. The system exhibits a strong waiting time dependence in the relaxation of both C and G, i.e., aging behavior at low temperatures. Moreover, the fluctuation-dissipation theorem (FDT), i.e., the relationship bewteen C and G in equilbrium, is modified in an interesting way. Similar FDT violation have been observed in the off-equilibrium dynamics of supercooled liquids in computer simulations[10, 11, 12] and an experiment.[13]

We note that the all the out of equilbrium glassy features in the p-spin and related models are driven, as in the equilibrium dynamics, by the dissipative nonlinearity in the equation of motion which comes from the nonlinear Hamiltonian. In the present toy model, as we see below, there is no dissipative nonlinearity since the Hamiltonian is trivial, i.e., gaussian without disorder. Instead the equation of motion involves the non-dissipative, i.e., reversible mode coupling nonliearities which drives the slowing down in the relaxation and the dynamic transition in the equilibrium dynamics. Our model possesses the reversible nonlinearities since we had a fluid in mind in constructing the model. Here we aim to see the out of equilibrium dynamics of the model driven by these reversible nonlinearities.

As a first step in this direction we derive below the self-consistent closed set of equations for five correlation functions and five response functions. The method used is standard: the generating functional method in which two fictitious external fields are introduced for each dynamical variables entering the model.[14] But in view of more complications involving correlation and response functions of these variables we will sketch a derivation. We indicate possibility of reducing these complicated equations to the set of two correlation functions and two response functions.

2. TOY MODEL

Here we consider the toy model with M-component velocity-like b variables and N-component density-like a variables, M being smaller than N. We have shown that in the limit of $M, N \to \infty$ with $\delta^* \equiv M/N$ finite, the parameter δ^* becomes a measure of hopping.[4] As a special case, if we take $\delta^* = 0$, then we obtain the zero-hopping model for the variables a, that is, the model is trivially non-ergodic. For $\delta^* = 1$ the hopping fully contributes and the system is always ergodic. For intermediate values of δ^* we expect an ergodic to nonergodic transition at some value of $T = T(\delta^*)$.

In this paper, after introducing the model we consider off-equilibrium dynamics that eventually involves 5 correlation functions and 5 response functions.

Our toy model is described by the variables a_j with $j = 1, 2, \ldots, N$, and b_α with $\alpha = 1, 2, \ldots, M$ which are sometimes abbreviated as $\hat{x}$ which spans the phase space of the model. The model is described by the following Langevin equation

$$
\begin{aligned}
\dot{a}_i &= K_{i\alpha} b_\alpha + \frac{\omega}{\sqrt{N}} J_{ij\alpha} a_j b_\alpha \\
\dot{b}_\alpha &= -\gamma b_\alpha - \omega^2 K_{j\alpha} a_j - \frac{\omega}{\sqrt{N}} J_{ij\alpha} (\omega^2 a_i a_j - T\delta_{ij}) + f_\alpha \\
\langle f_\alpha(t)\, f_\beta(t') \rangle &= 2\gamma T \delta_{\alpha\beta} \delta(t - t')
\end{aligned}
\tag{1}
$$

where the f's is the thermal noise with zero mean and the angular bracket is the thermal average over such noise, and the usual summation convention for repeated indices are used. Here and after we will use Roman indices for the component of a and Greek for that of b. Here γ gives a decay rate of the variable b_α and ω is seen to give a measure of the frequency of the oscillation of the variable a_j.

For later purpose, we require that the matrix $K_{i\alpha}$ satisfies $K_{i\alpha} K_{i\beta} = \delta_{\alpha\beta}$. It is then easy to show that the equilibrium stationary phase space distribution of the Fokker–Planck equation corresponding to (1) is given by[1, 2]

$$
\hat{D}_e(\hat{x}) \equiv cst\,.\, e^{-\Sigma_{j=1}^{N} \frac{\omega^2}{2T} a_j^2 - \Sigma_{\alpha=1}^{M} \frac{1}{2T} b_\alpha^2} \tag{2}
$$

where $cst.$ is understood to be a suitably chosen constant.

The mode-coupling coefficients $J_{ij\alpha}$ are considered to be *static* random variables satisfying the following statistical properties:

$$
\begin{aligned}
\overline{J_{ij\alpha}}^{J} &= 0, \\
\overline{J_{ij\alpha} J_{kl\beta}}^{J} &= \frac{g^2}{N} [(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\, \delta_{\alpha\beta} + K_{i\beta}(K_{k\alpha}\delta_{jl} + K_{l\alpha}\delta_{jk}) \\
&\quad + K_{j\beta}(K_{k\alpha}\delta_{il} + K_{l\alpha}\delta_{ik})]
\end{aligned}
\tag{3}
$$

where $\overline{\cdots}^{J}$ is the average over the independent Gaussian distribution of the J's. Eventually we take the mean field limit $M, N \to \infty$, keeping $\delta^* \equiv M/N$ finite.

In writing the model equation the temperature and other model parameters are fixed during time evolution, for example, in a situation after the quench. Naturally the model equation is valid only in such a time region.

3. ACTION INTEGRAL

In order to analyze the toy model in the limit $M, N \to \infty$, we introduce the following generating functional:[14]

$$Z\{h^a, \hat{h}^a, h^b, \hat{h}^b\} \equiv \int d\{a\} \int d\{b\} \int d\{\hat{a}\} \int d\{\hat{b}\} \times \exp\left\{i \int dt (h^a_j a_j + \hat{h}^a_j \hat{a}_j + h^b_\alpha b_\alpha + \hat{h}^b_\alpha \hat{b}_\alpha)\right\} \times e^{\mathscr{S}_0 + \mathscr{S}_I} \tag{4}$$

where

$$\hat{\mathscr{S}}_0 \equiv \int dt \{i\hat{a}_i(\dot{a}_i - K_{i\alpha} b_\alpha) + i\hat{b}_\alpha(\dot{b}_\alpha + \gamma b_\alpha + \omega^2 K_{i\alpha} a_i - f_\alpha)\}(t)$$

$$\hat{\mathscr{S}}_I \equiv J_{jk\alpha} X_{jk\alpha} \tag{5}$$

$$X_{jk\alpha} \equiv \int dt \frac{\omega}{\sqrt{N}} \{-i\hat{a}_j a_k b_\alpha + i\hat{b}_\alpha(\omega^2 a_j a_k - T\delta_{jk})\}(t) \tag{6}$$

In the above we have set the Jacobian of transformation of variables to unity assuming the Itô calculus.[4] In the limit $M, N \to \infty$, we find that the last term $-T\delta_{jk}$ in the integrand of (6) can be neglected, and will be dropped from now on. That is, we take

$$X_{jk\alpha} = \int dt \frac{\omega}{\sqrt{N}} \{-i\hat{a}_j a_k b_\alpha + i\omega^2 \hat{b}_\alpha a_j a_k\}(t) \tag{7}$$

[4] A consequence of choosing the Itô convention is the causality condition on the response functions. This implies that when the responses of $a(t)$ or $b(t)$ to the disturbances $\hat{a}(t')$ or $\hat{b}(t')$ occur simultaneously, the limit $t' \to t$ must be chosen in such a way that t is always *greater* than t'.

We then notice that the replacements $i\hat{a}_j \to (\omega^2/T)\, a_j,\ \hat{b}_\alpha \to b_\alpha/T$ on the rhs of (7) make this term vanish. Hence we can also rewrite (7) as

$$X_{jk\alpha} = \int dt\, \frac{\omega}{\sqrt{N}} \{-i\tilde{a}_j a_k b_\alpha + i\omega^2 \tilde{b}_\alpha a_j a_k\}(t) \tag{8}$$

where

$$i\tilde{a}_j \equiv i\hat{a}_j + \frac{\omega^2}{T}\, a_j, \qquad i\tilde{b}_\alpha \equiv i\hat{b}_\alpha + \frac{1}{T}\, b_\alpha. \tag{9}$$

The quantities of interest are the out of equilibrium correlation functions[5]

$$\begin{aligned}
C_a(t,t') &\equiv \frac{1}{N}\langle a_j(t)\, a_j(t')\rangle, & C_{ab}(t,t') &\equiv \frac{1}{M} K_{i\alpha}\langle a_i(t)\, b_\alpha(t')\rangle,\\
C_{ba}(t,t') &\equiv \frac{1}{M} K_{i\alpha}\langle b_\alpha(t)\, a_i(t')\rangle, & C_b(t,t') &\equiv \frac{1}{M}\langle b_\alpha(t)\, b_\alpha(t')\rangle\\
C_a^K(tt') &\equiv \frac{1}{M}\langle a_\alpha^K(t)\, a_\alpha^K(t')\rangle, & a_\alpha^K &\equiv K_{j\alpha} a_j
\end{aligned} \tag{10}$$

and the response functions

$$\begin{aligned}
G_a(t,t') &\equiv \frac{1}{N}\langle a_j(t)\, i\hat{a}_j(t')\rangle, & G_{ab}(t,t') &\equiv \frac{1}{M} K_{i\alpha}\langle a_i(t)\, i\hat{b}_\alpha(t')\rangle,\\
G_{ba}(t,t') &\equiv \frac{1}{M} K_{i\alpha}\langle b_\alpha(t)\, i\hat{a}_i(t')\rangle, & G_b(t,t') &\equiv \frac{1}{M}\langle b_\alpha(t)\, i\hat{b}_\alpha(t')\rangle,\\
G_a^K(tt') &\equiv \frac{1}{M}\langle a_\alpha^K(t)\, i\hat{a}_\alpha^K(t')\rangle, & i\hat{a}_\alpha^K &\equiv K_{j\alpha} i\hat{a}_j
\end{aligned} \tag{11}$$

We note that the new types of correlation and response functions C_a^K and G_a^K are needed to obtain the closed set of equations for C's and G's for $M < N$. For $M = N$ we have $C_a^K = C_a$ and $G_a^K = G_a$.

Since we are here concerned with out-of-equilibrium situation we will not use the time translation invariance nor the fluctuation-dissipation

[5] The definitions of C_{ab} and C_{ba} and those of the G's do not matter in the end. So we will use the symmetrically defined ones.

theorem (FDT).[6] We now take averages of (6) over f_α and the J's where we use

$$\begin{aligned}\langle e^{-i\int dt\,\hat{b}_\alpha(t)\,f_\alpha(t)}\rangle &= e^{-\gamma T\int dt\,\hat{b}_\alpha(t)^2}\\ \overline{e^{J_{jk\alpha}X_{jk\alpha}}}^{J} &= e^{\frac{1}{2}\overline{J_{jk\alpha}J_{lm\beta}}^{J}X_{jk\alpha}X_{lm\beta}}\end{aligned} \tag{12}$$

Then we have

$$\langle e^{\hat{\mathscr{S}}_0}\rangle \equiv e^{\mathscr{S}_0}, \qquad \overline{e^{\hat{\mathscr{S}}_I}}^{J} \equiv e^{\mathscr{S}_I} \tag{13}$$

where

$$\begin{aligned}\mathscr{S}_0 &\equiv \int dt\{i\hat{a}_i(\dot{a}_i - K_{i\alpha}b_\alpha) + i\hat{b}_\alpha(\dot{b}_\alpha + \gamma b_\alpha + \omega^2 K^T_{\alpha i}a_i + \gamma T i\hat{b}_\alpha)\}(t),\\ \mathscr{S}_I &\equiv \frac{g^2}{N}\int dt\int dt'\,\theta(t-t')\,\hat{\xi}_{ij\alpha}(t)[(\tilde{\xi}_{ij\alpha}(t') + \tilde{\xi}_{ji\alpha}(t'))\\ &\quad + K_{i\beta}K_{k\alpha}(\tilde{\xi}_{kj\beta}(t') + \tilde{\xi}_{jk\beta}(t')) + K_{j\beta}K_{k\alpha}(\tilde{\xi}_{ki\beta}(t') + \tilde{\xi}_{ik\beta}(t'))]\end{aligned} \tag{14}$$

[6] The usual FDT takes the form where the response function is proportional to the time derivative of the correlation function. But here in the case of gaussian Hamiltonian[(15)] the FDT is given by

$$G_a(t-t') = -\theta(t-t')\frac{\omega^2}{T}C_a(t-t'), \qquad G_{ab}(t-t') = -\theta(t-t')\frac{1}{T}C_{ab}(t-t'),$$

$$G_{ba}(t-t') = -\theta(t-t')\frac{\omega^2}{T}C_{ba}(t-t'), \qquad G_b(t-t') = -\theta(t-t')\frac{1}{T}C_b(t-t'),$$

$$G^K_a(t-t') = -\theta(t-t')\frac{\omega^2}{T}C^K_a(t-t')$$

where $\theta(t)$ is the usual step function equal to 1 for positive t and zero otherwise, the appearance of which comes from the causality. Another property arising from the causality plus the above FDT is the following for arbitrary $X(t) = X(a(t), b(t), \hat{a}(t), \hat{b}(t))$:

$$\langle \hat{A}(t)\,X(t')\rangle = \langle X(t)\,\tilde{A}(t')\rangle = 0 \qquad \text{for} \quad t > t'$$

where $A(t) = (a(t), b(t))$ and the indices are suppressed for brevity. This fact is only limited to equilibrium when $h = \hat{h} = 0$. Hence $\langle X(t)\,\tilde{A}(t')\rangle = 0$ for $t > t'$ will not be used here. For $\hat{h} = 0$ and arbitrary h, however, the causality requires $\langle \hat{A}(t)\,X(t')\rangle = 0$ for $t > t'$, which will be used later.

and

$$
\begin{aligned}
\hat{\xi}_{ij\alpha}(t) &\equiv \hat{\xi}^{a}_{ij\alpha}(t)+\hat{\xi}^{b}_{ij\alpha}(t), & \tilde{\xi}_{ij\alpha}(t') &\equiv \tilde{\xi}^{a}_{ij\alpha}(t')+\tilde{\xi}^{b}_{ij\alpha}(t') \\
\hat{\xi}^{a}_{ij\alpha}(t) &\equiv \frac{\omega}{\sqrt{N}}(-i\hat{a}_i a_j b_\alpha)(t), & \hat{\xi}^{b}_{ij\alpha}(t) &\equiv \frac{\omega^3}{\sqrt{N}}(i\hat{b}_\alpha a_i a_j)(t) \\
\tilde{\xi}^{a}_{ij\alpha}(t') &\equiv \frac{\omega}{\sqrt{N}}(-i\tilde{a}_i a_j b_\alpha)(t'), & \tilde{\xi}^{b}_{ij\alpha}(t') &\equiv \frac{\omega^3}{\sqrt{N}}(i\tilde{b}_\alpha a_i a_j)(t').
\end{aligned}
\tag{15}
$$

Here we remind that we can interchange $\hat{\xi}_{ij\alpha}$ and $\tilde{\xi}_{ij\alpha}$ in $\hat{\mathscr{S}}_I$, (6), as we have noted in connection with (8) and (9). Therefore we can split $\mathscr{S}_I$ into the 4 contributions

$$
\mathscr{S}_I \equiv \mathscr{S}^{aa}_I+\mathscr{S}^{ab}_I+\mathscr{S}^{ba}_I+\mathscr{S}^{bb}_I. \tag{16}
$$

where

$$
\begin{aligned}
\mathscr{S}^{aa}_I \equiv{}& \frac{g^2}{N}\int dt\int dt'\;\theta(t-t')\;\hat{\xi}^{a}_{ij\alpha}(t)[\overbrace{\tilde{\xi}^{a}_{ij\alpha}(t')}+\tilde{\xi}^{a}_{ji\alpha}(t') \\
&+K_{i\beta}K_{k\alpha}(\tilde{\xi}^{a}_{kj\beta}(t')+\tilde{\xi}^{a}_{jk\beta}(t'))+K_{j\beta}K_{k\alpha}(\tilde{\xi}^{a}_{ki\beta}(t')+\overbrace{\tilde{\xi}^{a}_{ik\beta}(t')})], \\
\mathscr{S}^{ab}_I \equiv{}& \frac{g^2}{N}\int dt\int dt'\theta(t-t')\;\hat{\xi}^{a}_{ij\alpha}(t)[(\tilde{\xi}^{b}_{ij\alpha}(t')+\tilde{\xi}^{b}_{ji\alpha}(t')) \\
&+K_{i\beta}K_{k\alpha}(\overbrace{\tilde{\xi}^{b}_{kj\beta}(t')}+\overbrace{\tilde{\xi}^{b}_{jk\beta}(t')})+K_{j\beta}K_{k\alpha}(\tilde{\xi}^{b}_{ki\beta}(t')+\tilde{\xi}^{b}_{ik\beta}(t'))], \\
\mathscr{S}^{ba}_I \equiv{}& \frac{g^2}{N}\int dt\int dt'\theta(t-t')\;\hat{\xi}^{b}_{ij\alpha}(t)[(\tilde{\xi}^{a}_{ij\alpha}(t')+\tilde{\xi}^{a}_{ji\alpha}(t')) \\
&+K_{i\beta}K_{k\alpha}(\overbrace{\tilde{\xi}^{a}_{kj\beta}(t')}+\tilde{\xi}^{a}_{jk\beta}(t'))+K_{j\beta}K_{k\alpha}(\overbrace{\tilde{\xi}^{a}_{ki\beta}(t')}+\tilde{\xi}^{a}_{ik\beta}(t'))], \\
\mathscr{S}^{bb}_I \equiv{}& \frac{g^2}{N}\int dt\int dt'\theta(t-t')\;\hat{\xi}^{b}_{ij\alpha}(t)[(\overbrace{\tilde{\xi}^{b}_{ij\alpha}(t')}+\overbrace{\tilde{\xi}^{b}_{ji\alpha}(t')}) \\
&+K_{i\beta}K_{k\alpha}(\tilde{\xi}^{b}_{kj\beta}(t')+\tilde{\xi}^{b}_{jk\beta}(t'))+K_{j\beta}K_{k\alpha}(\tilde{\xi}^{b}_{ki\beta}(t')+\tilde{\xi}^{b}_{ik\beta}(t'))]
\end{aligned}
\tag{17}
$$

Here the terms giving non-vanishing contributions in the equilibrium were overbraced. But this is no longer enough in out of equilibrium situation as we shall see. As an illustration we look at the first term of (17) or its integrand $\mathscr{S}^{aa}_I$. That is

$$
\begin{aligned}
\hat{\xi}^{a}_{ij\alpha}(t)\;\tilde{\xi}^{a}_{ij\alpha}(t') &= \frac{\omega^2}{N}[-i\hat{a}_i(t)\;a_j(t)\;b_\alpha(t)][-i\tilde{a}_i(t')\;a_j(t')\;b_\alpha(t')] \\
&= \omega^2 N\delta^* i\hat{a}_i(t)\;i\tilde{a}_i(t')\;C_a(tt')\;C_b(tt')+\cdots
\end{aligned}
\tag{18}
$$

where the ellipsis contains quantities like $\langle i\hat{a}_i(t)\cdots\rangle$ which are absent for $h, \hat{h}=0$ and also contains other fluctuation terms. Due to the presence of a factor N in front of the second member of the rhs of (18), these fluctuation terms disappear in the limit $M, N \to \infty$ with a finite δ^*. We then analyze each factor like $\hat{\xi}^a_{ij\alpha}(t)\,\tilde{\xi}^a_{ij\alpha}(t')$ in (17) in the limit $M, N \to \infty$, where we only impose the causality condition $t \geqslant t'$ reflected by $\theta(t-t')$ in (17) but not FDT.

After tedious but straightforward algebra with $h=0$ we arrive at the effective quadratic action given below. To simplify the expression we introduce the following notation:[7]

$$X \otimes Y(t) = \int_{-\infty}^{t} dt'\; X(tt')\, Y(t')$$

In writing down the quadratic action below, for simplicity, we suppress time arguments and indices for the variables a, b, so that for instance we write $\mathbf{K}\cdot\mathbf{b}$ for $K_{j\alpha} b_\alpha$ in matrix notation. We also omit integral signs and $\otimes$ for the moment. The total action is given by the following matrix form

$$\begin{aligned}
\mathscr{S}_{tot} &= \hat{\mathscr{S}}_{eq} + \mathscr{S}_{eq} + \mathscr{S}_{oe} \\
&\equiv (i\hat{\mathbf{a}}, i\hat{\mathbf{b}}) \cdot \hat{\Omega}_{eq} \cdot \begin{pmatrix} i\hat{\mathbf{a}} \\ i\hat{\mathbf{b}} \end{pmatrix} + (i\hat{\mathbf{a}}, i\hat{\mathbf{b}}) \cdot \Omega_{eq} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \\
&\quad + (i\hat{\mathbf{a}}, i\hat{\mathbf{b}}) \cdot \Omega^{I}_{oe} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}
\end{aligned} \tag{19}$$

where [eq] and [oe] stand for equilibrium and off-equilibriun, respectively. Also note $\hat{\mathscr{S}}_{oe}$ is absent. We have

$$\hat{\Omega}_{eq} \equiv \hat{\Omega}_0 + \hat{\Omega}_I \tag{20}$$

$$\hat{\Omega}_0 \equiv \begin{pmatrix} \mathbf{0}_N & \mathbf{0}_{NM} \\ \mathbf{0}_{MN} & \gamma T \mathbf{1}_M \end{pmatrix} \tag{21}$$

$$\hat{\Omega}_I \equiv \begin{pmatrix} \mathbf{1}_N \frac{T}{\omega^2} \Sigma_{aa} & T\Sigma_{ab}\mathbf{K} \\ \frac{T}{\omega^2} \Sigma_{ba}\mathbf{K}^T & \mathbf{1}_M T \Sigma_{bb} \end{pmatrix} \tag{22}$$

[7] Note $\otimes$ signifies causality. Alternatively, this causality is taken care of by redefining the Σ's by absorbing $\theta(t-t')$ into them. Then the above integral is from $-\infty$ to ∞. If desired one can do the same for the response functions.

and furthermore,

$$\Omega_{eq} = \Omega^0_{eq} + \Omega^I_{eq} \tag{23}$$

$$\Omega^0_{eq} \equiv \begin{pmatrix} \mathbf{1}_N \partial_t & -\mathbf{K} \\ \omega^2 \mathbf{K}^T & \mathbf{1}_M(\partial_t + \gamma) \end{pmatrix} \tag{24}$$

$$\Omega^I_{eq} \equiv \begin{pmatrix} \mathbf{1}_N \Sigma_{aa} & \mathbf{K}\Sigma_{ab} \\ \Sigma_{ba}\mathbf{K}^T & \mathbf{1}_M \Sigma_{bb} \end{pmatrix} \tag{25}$$

and

$$\Omega^I_{oe} \equiv \begin{pmatrix} \mathbf{1}_N(-\Sigma_{aa} + \Delta\Sigma_{aa} - 2\Delta\Sigma^{\odot}_{aa}) & -(\Sigma_{ab} + \Delta\Sigma_{ab})\,\mathbf{K} \\ (-\Sigma_{ba} + \Delta\Sigma_{ba} - 2\Delta\Sigma^{\odot}_{ba})\,\mathbf{K}^T & -\mathbf{1}_M(\Sigma_{bb} + \Delta\Sigma_{bb}) \end{pmatrix} \tag{26}$$

where $\mathbf{1}_N(\mathbf{1}_M)$ are the unit matrix of rank $N(M)$, $\mathbf{K}$ is the $N \times M$ matrix with the elements $K_{j\alpha}$ and $\mathbf{K}^T$ its transposed $M \times N$ matrix. All these matrices are multiplied by a matrix whose tt' element is the delta function $\delta(t-t')$. Also $\mathbf{0}_{NM}$, $\mathbf{0}_{MN}$ are the 0 matrices of $N \times M$, $M \times N$, respectively. Here the memory kernel Σ's are defined by

$$\Sigma_{aa}(tt') \equiv \frac{g^2\omega^4}{T}\left(\delta^* C_a(tt')\, C_b(tt') + (\delta^*)^2\, C_{ab}(tt')\, C_{ba}(tt')\right)$$

$$\Sigma_{ab}(tt') \equiv -2\frac{g^2\omega^4}{T}\,\delta^* C_a(tt')\, C_{ba}(tt'),$$

$$\Sigma_{ba}(tt') \equiv -2\frac{g^2\omega^6}{T}\,\delta^* C_a(tt')\, C_{ab}(tt'),$$

$$\Sigma_{bb}(tt') \equiv \frac{2g^2\omega^6}{T}\, C_a(tt')^2 \tag{27}$$

The other types of kernels $\Delta\Sigma$ and $\Delta\Sigma^{\odot}$ are defined as follows:

$$\Delta\Sigma_{aa} \equiv g^2\omega^2(\delta^* G_a C_b + (\delta^*)^2\, C_{ab} G_{ba}) \tag{28}$$

$$\Delta\Sigma^{\odot}_{aa} \equiv g^2\omega^4(\delta^* C_a G_b + (\delta^*)^2\, C_{ba} G_{ab}) \tag{29}$$

$$\Delta\Sigma_{ab} \equiv -g^2\omega^2\delta^*[C_a G_{ba} + G_a C_{ba}] \tag{30}$$

$$\Delta\Sigma_{bb} \equiv 2g^2\omega^4 C_a G_a, \qquad \Delta\Sigma_{ba} \equiv -2g^2\omega^4\delta^* C_{ab} G_a \tag{31}$$

$$\Delta\Sigma^{\odot}_{ba} \equiv -2g^2\omega^6\delta^* C_a G_{ab} \tag{32}$$

Note that every term of the $\Delta\Sigma$'s contains one factor of the G's and one factor of C's in contrast to the Σ's, (27), each of which contains the two factors of C's.

4. CORRELATION AND RESPONSE FUNCTIONS

We now proceed to response and correlation functions. We first introduce correlation and response matrices **C** and **G** with sub-matrices $\mathbf{C}_{aa} \equiv \langle \mathbf{aa} \rangle$ and $\mathbf{G}_{aa} \equiv \langle \mathbf{a} i \hat{\mathbf{a}} \rangle$ etc. whose elements are $\langle \mathbf{aa} \rangle_{it,\, jt'} = \langle a_i(t)\, a_j(t') \rangle$ etc. Thus the entire correlation and response matrices are written as

$$\mathbf{C} \equiv \begin{pmatrix} \mathbf{C}_{aa} & \mathbf{C}_{ab} \\ \mathbf{C}_{ba} & \mathbf{C}_{bb} \end{pmatrix}, \qquad \mathbf{G} \equiv \begin{pmatrix} \mathbf{G}_{aa} & \mathbf{G}_{ab} \\ \mathbf{G}_{ba} & \mathbf{G}_{bb} \end{pmatrix} \tag{33}$$

The formal matrix equations determining correlation and response matrices take the form which are obtained from the effective action defined through (19) to (26):

$$\Omega \cdot \mathbf{G} = \mathbf{1} \tag{34}$$

$$\Omega \cdot \mathbf{C} = (\hat{\Omega}_{eq} + \hat{\Omega}^{\dagger}_{eq}) \cdot \mathbf{G}^{\dagger} \tag{35}$$

$$\Omega \equiv \Omega_{eq} + \Omega^{I}_{oe} \tag{36}$$

4.1. Response Function

We first take up the response functions. The equations for them are written in terms of submatrices as follows where equilibrium and off-equilibrium parts are separated:

$$\begin{aligned} &(\partial_t + \Sigma_{aa})\, \mathbf{G}_{aa} - (1 - \Sigma_{ab})\, \mathbf{K} \cdot \mathbf{G}_{ba} \\ &+ [\![(-\Sigma_{aa} + \Delta\Sigma_{aa} - 2\Delta\Sigma^{\odot}_{aa})\, \mathbf{G}_{aa} - (\Sigma_{ab} + \Delta\Sigma_{ab})\, \mathbf{K} \cdot \mathbf{G}_{ba})]\!] \\ &= \mathbf{1}_N \end{aligned} \tag{37}$$

$$\begin{aligned} &(\partial_t + \Sigma_{aa})\, \mathbf{G}_{ab} - (1 - \Sigma_{ab})\, \mathbf{K} \cdot \mathbf{G}_{bb} \\ &+ [\![(-\Sigma_{aa} + \Delta\Sigma_{aa} - 2\Delta\Sigma^{\odot}_{aa})\, \mathbf{G}_{ab} - (\Sigma_{ab} + \Delta\Sigma_{ab})\, \mathbf{K} \cdot \mathbf{G}_{bb})]\!] \\ &= \mathbf{0}_{NM} \end{aligned} \tag{38}$$

$$\begin{aligned} &(\omega^2 + \Sigma_{ba})\, \mathbf{K}^T \cdot \mathbf{G}_{aa} + (\partial_t + \gamma + \Sigma_{bb})\, \mathbf{G}_{ba} \\ &+ [\![(-\Sigma_{ba} + \Delta\Sigma_{ba} - 2\Delta\Sigma^{\odot}_{ba})\, \mathbf{K}^T \cdot \mathbf{G}_{aa} - (\Sigma_{bb} + \Delta\Sigma_{bb}) \cdot \mathbf{G}_{ba}]\!] \\ &= \mathbf{0}_{MN} \end{aligned} \tag{39}$$

$$\begin{aligned} &(\omega^2 + \Sigma_{ba})\, \mathbf{K}^T \cdot \mathbf{G}_{ab} + (\partial_t + \gamma + \Sigma_{bb})\, \mathbf{G}_{bb} \\ &+ [\![(-\Sigma_{ba} + \Delta\Sigma_{ba} - 2\Delta\Sigma^{\odot}_{ba})\, \mathbf{K}^T \cdot \mathbf{G}_{ab} - (\Sigma_{bb} + \Delta\Sigma_{bb}) \cdot \mathbf{G}_{bb}]\!] \\ &= \mathbf{1}_M \end{aligned} \tag{40}$$

Here $[\![\cdots]\!]$ are off-equilibrium parts which vanish in equilibrium due to the FDT which makes all the $\Delta\Sigma$'s and $\Delta\Sigma^{\odot}$ equal to $-\Sigma$'s. From this we can deduce the equations for 5 response functions in the following manner.

We define the following notation valid for arbitrary matrix $\mathbf{X}, \mathbf{Y}$ of ranks N, M, respectively:

$$\mathrm{tr}^a\, \mathbf{X} \equiv \frac{1}{N} \sum_j X_{jj}, \qquad \mathrm{tr}^b\, \mathbf{Y} \equiv \frac{1}{M} \sum_\alpha Y_{\alpha\alpha}$$

We first apply $\mathrm{tr}^a \cdots$ to (37), next $\mathrm{tr}^b\, \mathbf{K}^T \cdots$ to (38), then $\mathrm{tr}^a\, \mathbf{K} \cdots$ to (39), also $\mathrm{tr}^b \cdots$ to (40), and finally $\mathrm{tr}^b\, \mathbf{K}^T \cdots \mathbf{K}$ to (37).

We then end up with the following set of 5 equations for 5 response functions where again equilibrium and off-equilibrium parts are separated:

$$\begin{aligned} &[(\partial_t + \Sigma_{aa})\, G_a - (1 - \Sigma_{ab})\, \delta^* G_{ba}](tt') \\ &+ [\![(-\Sigma_{aa} + \Delta\Sigma_{aa} - 2\Delta\Sigma^{\odot}_{aa})\, G_a - (\Sigma_{ab} + \Delta\Sigma_{ab})\, \delta^* G_{ba}]\!](tt') \\ &= \delta(t - t') \end{aligned} \tag{41}$$

$$\begin{aligned} &[(\partial_t + \Sigma_{aa})\, G_{ab} - (1 - \Sigma_{ab})\, G_b](tt') \\ &+ [\![(-\Sigma_{aa} + \Delta\Sigma_{aa} - 2\Delta\Sigma^{\odot}_{aa})\, G_{ab} - (\Sigma_{ab} + \Delta\Sigma_{ab})\, G_b]\!](tt') \\ &= 0 \end{aligned} \tag{42}$$

$$\begin{aligned} &[(\omega^2 + \Sigma_{ba})\, G^K_a + (\partial_t + \gamma + \Sigma_{bb})\, G_{ba}](tt') \\ &+ [\![(-\Sigma_{ba} + \Delta\Sigma_{ba} - 2\Delta\Sigma^{\odot}_{ba})\, G^K_a - (\Sigma_{bb} + \Delta\Sigma_{bb})\, G_{ba}]\!](tt') \\ &= 0 \end{aligned} \tag{43}$$

$$\begin{aligned} &[(\omega^2 + \Sigma_{ba})\, G_{ab} + (\partial_t + \gamma + \Sigma_{bb})\, G_b](tt') \\ &+ [\![(-\Sigma_{ba} + \Delta\Sigma_{ba} - 2\Delta\Sigma^{\odot}_{ba})\, G_{ab} - (\Sigma_{bb} + \Delta\Sigma_{bb})\, G_b]\!](tt') \\ &= \delta(t - t') \end{aligned} \tag{44}$$

$$\begin{aligned} &[(\partial_t + \Sigma_{aa})\, G^K_a - (1 - \Sigma_{ab})\, G_{ba}](tt') \\ &+ [\![(-\Sigma_{aa} + \Delta\Sigma_{aa} - 2\Delta\Sigma^{\odot}_{aa})\, G^K_a - (\Sigma_{ab} + \Delta\Sigma_{ab})\, G_{ba}]\!](tt') \\ &= \delta(t - t') \end{aligned} \tag{45}$$

4.2. Correlation Functions

We start with the formula (35) where the lhs are the same as those of (37) to (40) except that the $\mathbf{G}$'s are replaced by the $\mathbf{C}$'s. Thus we need to consider only the rhs. Note that the rhs is the same as the equilibrium case if $\mathbf{G}^{\dagger}(tt')$ is given. $\mathbf{G}^{\dagger}(tt')$ in terms of submatrices is given by

$$\mathbf{G}^{\dagger}(tt') = \begin{pmatrix} \langle i\hat{\mathbf{a}}(t')\, \mathbf{a}(t)\rangle & \langle i\hat{\mathbf{a}}(t')\, \mathbf{b}(t)\rangle \\ \langle i\hat{\mathbf{b}}(t')\, \mathbf{a}(t)\rangle & \langle i\hat{\mathbf{b}}(t')\, \mathbf{b}(t)\rangle \end{pmatrix} \tag{46}$$

Also one can work out

$$\begin{aligned}
&(\hat{\Omega}_{eq} + \hat{\Omega}^{\dagger}_{eq})(tt') \\
&= \begin{pmatrix} \frac{T}{\omega^2}[\Sigma_{aa}(tt') + \Sigma_{aa}(t't)]\, \mathbf{1}_N, & T[\Sigma_{ab}(tt') + \frac{1}{\omega^2}\Sigma_{ba}(t't)]\, \mathbf{K} \\ T[\frac{1}{\omega^2}\Sigma_{ba}(tt') + \Sigma_{ab}(t't)]\, \mathbf{K}^T, & T[2\gamma\delta(t-t') + \Sigma_{bb}(tt') + \Sigma_{bb}(t't)]\, \mathbf{1}_M \end{pmatrix} \\
&= \begin{pmatrix} 2\frac{T}{\omega^2}\Sigma_{aa}(tt')\, \mathbf{1}_N, & 2T\Sigma_{ab}(tt')\, \mathbf{K} \\ 2T\Sigma_{ab}(t't)\, \mathbf{K}^T, & 2T[\gamma\delta(t-t') + \Sigma_{bb}(tt')]\, \mathbf{1}_M \end{pmatrix}
\end{aligned} \tag{47}$$

The last equality in (47) is due to the following symmetric properties of Σ's under exchange of two times: $\Sigma_{aa}(t, t') = \Sigma_{aa}(t', t)$, $\Sigma_{ab}(t, t') = \frac{1}{\omega^2}\Sigma_{ba}(t', t)$, and $\Sigma_{bb}(t, t') = \Sigma_{bb}(t, 't)$. These follow directly from $C_a(t, t) = C_a(t', t)$, $C_b(t, t) = C_b(t', t)$, and $C_{ab}(t, t') = C_{ba}(t', t)$. The matrix equation (35) can be split into 4 submatrix equations as follows where the lhs is abbrevitated as $\partial_t \mathbf{C}_{aa}(tt') + \cdots$ etc.

$$\begin{aligned}
\partial_t \mathbf{C}_{aa}(tt') + \cdots &= 2\frac{T}{\omega^2}\Sigma_{aa}(t\bullet)\langle i\hat{\mathbf{a}}(\bullet)\, \mathbf{a}(t')\rangle \\
&\quad + 2T\Sigma_{ab}(t\bullet)\, \mathbf{K}\cdot\langle i\hat{\mathbf{b}}(\bullet)\, \mathbf{a}(t')\rangle,
\end{aligned} \tag{48}$$

$$\begin{aligned}
\partial_t \mathbf{C}_{ab}(tt') + \cdots &= 2\frac{T}{\omega^2}\Sigma_{aa}(t\bullet)\langle i\hat{\mathbf{a}}(\bullet)\, \mathbf{b}(t')\rangle \\
&\quad + 2T\Sigma_{ab}(t\bullet)\, \mathbf{K}\cdot\langle i\hat{\mathbf{b}}(\bullet)\, \mathbf{b}(t')\rangle,
\end{aligned} \tag{49}$$

$$\begin{aligned}
\partial_t \mathbf{C}_{ba}(tt') + \cdots &= 2T\Sigma_{ab}(\bullet t)\, \mathbf{K}^T\cdot\langle i\hat{\mathbf{a}}(\bullet)\, \mathbf{a}(t')\rangle \\
&\quad + 2T[\gamma\delta(t-\bullet) + \Sigma_{bb}(t\bullet)]\langle i\hat{\mathbf{b}}(\bullet)\, \mathbf{a}(t')\rangle,
\end{aligned} \tag{50}$$

$$\begin{aligned}
\partial_t \mathbf{C}_{bb}(tt') + \cdots &= 2T\Sigma_{ab}(\bullet t)\, \mathbf{K}^T\cdot\langle i\hat{\mathbf{a}}(\bullet)\, \mathbf{b}(t')\rangle \\
&\quad + 2T[\gamma\delta(t-\bullet) + \Sigma_{bb}(t\bullet)]\langle i\hat{\mathbf{b}}(\bullet)\, \mathbf{b}(t')\rangle
\end{aligned} \tag{51}$$

The rhs are the same with the equilibrium case.

We are ready to find the rhs of the equations for correlation functions, which can be done following the same procedure as for the G's. Thus we first apply $\mathrm{tr}^a \cdots$ to (48), next $\mathrm{tr}^b\, \mathbf{K}^T \cdots$ to the rhs of (49), then $\mathrm{tr}^a\, \mathbf{K} \cdots$ to the rhs of (50), also $\mathrm{tr}^b \cdots$ to the rhs of (51), and finally $\mathrm{tr}^b\, \mathbf{K}^T \cdots \mathbf{K}$ to the rhs of (48).

Results are summarized below in the form of 5 self-consistent equations for 5 correlation functions:

$$
\begin{aligned}
&[(\partial_t + \Sigma_{aa})\, C_a - (1-\Sigma_{ab})\, \delta^* C_{ba}](tt') \\
&+ [\![(-\Sigma_{aa} + \Delta\Sigma_{aa} - 2\Delta\Sigma^{\odot}_{aa})\, C_a - (\Sigma_{ab} + \Delta\Sigma_{ab})\, \delta^* C_{ba}]\!](tt') \\
&= 2\frac{T}{\omega^2}\, \Sigma_{aa}(t\bullet)\, G_a(t'\bullet) + 2T\Sigma_{ab}(t\bullet)\, \delta^* G_{ab}(t'\bullet),
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
&[(\partial_t + \Sigma_{aa})\, C_{ab} - (1-\Sigma_{ab})\, C_b](tt') \\
&+ [\![(-\Sigma_{aa} + \Delta\Sigma_{aa} - 2\Delta\Sigma^{\odot}_{aa})\, C_{ab} - (\Sigma_{ab} + \Delta\Sigma_{ab})\, C_b]\!](tt') \\
&= 2\frac{T}{\omega^2}\, \Sigma_{aa}(t\bullet)\, G_{ba}(t'\bullet) + 2T\Sigma_{ab}(t\bullet)\, G_b(t'\bullet),
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
&[(\omega^2 + \Sigma_{ba})\, C^K_a + (\partial_t + \gamma + \Sigma_{bb})\, C_{ba}](tt') \\
&+ [\![(-\Sigma_{ba} + \Delta\Sigma_{ba} - 2\Delta\Sigma^{\odot}_{ba})\, C^K_a - (\Sigma_{bb} + \Delta\Sigma_{bb})\, C_{ba}]\!](tt') \\
&= 2T\Sigma_{ab}(\bullet t)\, G^K_a(t'\bullet) + 2T[\gamma\delta(t-\bullet) + \Sigma_{bb}(t\bullet)]\, G_{ab}(t'\bullet),
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
&[(\omega^2 + \Sigma_{ba})\, C_{ab} + (\partial_t + \gamma + \Sigma_{bb})\, C_b](tt') \\
&+ [\![(-\Sigma_{ba} + \Delta\Sigma_{ba} - 2\Delta\Sigma^{\odot}_{ba})\, C_{ab} - (\Sigma_{bb} + \Delta\Sigma_{bb})\, C_b]\!](tt') \\
&= 2T\Sigma_{ab}(\bullet t)]\, G_{ba}(t'\bullet) + 2T[\gamma\delta(t-\bullet) + \Sigma_{bb}(t\bullet)]\, G_b(t'\bullet),
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
&[(\partial_t + \Sigma_{aa})\, C^K_a - (1-\Sigma_{ab})\, C_{ba}](tt') \\
&+ [\![(-\Sigma_{aa} + \Delta\Sigma_{aa} - 2\Delta\Sigma^{\odot}_{aa})\, C^K_a - (\Sigma_{ab} + \Delta\Sigma_{ab})\, C_{ba}]\!](tt') \\
&= 2\frac{T}{\omega^2}\, \Sigma_{aa}(t\bullet)\, G^K_a(t'\bullet) + 2T\Sigma_{ab}(t\bullet)\, G_{ab}(t'\bullet)
\end{aligned} \tag{56}
$$

The 5 equations (41)–(45) and 5 equations (52)–(56) constitute 10 equations that self-consistently determine 5 correlation functions and 5 response functions. Note the rhs of (52)–(56) are the same as in equilibrium case. The lhs again have been split up into equilibrium and nonequilibrium portions as we have done in (41)–(45).

We now give several comments related to the set of equations (52)–(56):

- So far we have considered a general out of equilibrium situation where the model parameter remains constant. Now we specify the condition to meet a typical aging experiment. We then suppose that a system in some equilibrium state is quenched at the time $t=0$ and starts to evolve into another equilibrium state which characterizes the model parameters. We let the system age till some time $t_w(>0)$, and we measure at a later time $t(>t_w)$. This would mean that in the above equations we should take $t'=t_w$ and the time integrals denoted by $\bullet \equiv s$ should be over the region $s>0$, which is further limited by the causality conditions on the Σ's and the G's. Thus each integral in the equations of the response functions, (41)–(45), is in the interval $t_w<s<t$. Similarly, each integral in the lhs of the equations for the correlation functions, (52)–(56), is in the interval $0<s<t$ whereas each integral in the rhs is in the interval $0<s<t_w$. We then observe that the rhs of (52) to (56) are the source of contributions to the correlation functions from thermal noise generated after the quench. Note that the functions that multiply the $G(t'\bullet)$'s in the rhs of (52) to (56) are the correlation functions of renormalized thermal noises. See the equations (25) of ref. 2.

- In the discussion above we have not considered effects of the initial condition at the time of quench, say, t_0.[16] This can be studied by inserting the properly normalized weight factor proportional to $\exp(-H/k_BT_0)$ where H is the system Hamiltonian and T_0 is the initial temperature. For spin glass cases including Potts or p-spin systems, H contains quenched random interaction parameters, which have to be included in averaging the exponential of the action integral over such quenched random parameters contained in the action integral. This results in additional terms in the final effective action. In our toy model, this complication is absent because the Hamiltonian is free from quenched disorder. The initial condition enters only through the initial values of the correlation functions, that is, $C_a(t=0, t_w=0)=T_0/\omega^2$, etc.

- If we take all the $\Delta\Sigma$'s and $\Delta\Sigma^{\odot}$ equal to $-\Sigma$'s, consequences of FDT, we recover the equilibrium equations. In this case the rhs of 5 equations (52)–(56) must vanish for $t>t'$ because of the FDT which tells that the G's to be proportional to the C's. The direct verification of this follows if we note that in equilibrium t_w can be shifted to 0. This makes the integration interval of s, and hence the rhs of (52) to (56) to vanish.

- No $\Delta\Sigma^{\odot}_{ab}$ appears in constrast to $\Delta\Sigma^{\odot}_{ba}$, which is due to asymmetric way a and b variables enter dynamics.

- No C^K_a and G^K_a appear in the Σ's. One can see this by inspecting the derivations of (19) to (26). We see that no terms containing the combinations $a^K_\alpha(t)\, a^K_\alpha(t')$, $a^K_\alpha(t)\, i\hat{a}^K_\alpha(t')$ with $a^K_\alpha \equiv K_{j\alpha}a_j$, $i\hat{a}^K_\alpha \equiv K_{j\alpha}i\hat{a}_j$ appear.

5. CONCLUDING REMARKS

In the foregoing sections we have derived the exact self-consistent equations for correlation functions and response functions for our toy model which, in some sense, is complementary to the works of Latz[17] who studied the microscopic fluid system.

The set of ten equations of our self-consistent scheme would be too complicated for further analyses at this time. Now, the velocity-like b-variables can be made rapidly decaying by choosing a sufficiently large value for γ. In particular, this gives rise to a possibility of adiabatically eliminating the velocity-like variables as we have done to derive a Fokker–Planck type equation for the probability distribution function of a-variables only in refs. 2 and 3. We have recently examined this problem for the simpler equilibrium case of refs. 2 and 3. Even in this case we are finding subtle points that require a multiple-time-scale analysis.[18] Although our ultimate aim is to analyze the behavior of the equations found in this paper, their complexity requires us to first look at limiting cases with possible simplifications of the equations. This is deferred to future investigations.

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